

The Asymptotic Mandelbrot Law of Some Evolution Networks

Li Li

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Abstract

In this letter, we study some evolution networks that grow with linear preferential attachment. Based upon some recent results on the quotient Gamma function, we give a rigorous proof of the asymptotic Mandelbrot law for the degree distribution $p_k \propto (k + c)^{-\gamma}$ in certain conditions. We also analytically derive the best fitting values for the scaling exponent γ and the shifting coefficient c .

Complex networks are now the joint focus of many branches of research^[1–3]. Particularly, the scale-free property of some networks attracts continuous interests, due to their importance and pervasiveness^[4–6]. In short, this property means that the degree distribution of a network obeys a power law $P(k) \propto k^{-\gamma}$, where k is the degree and $P(k)$ is the corresponding probability density, and the scaling exponent γ is a constant. A pioneering model that generates power-law degree distribution was presented by Barabási and Albert (BA)^[4].

In recent studies, it was found that in some complex networks, e.g. transportation networks^[7] and social collaboration networks^[8], the degree distribution follows the so-called “shifted power law”^[9] $P(k) \propto (k + c)^{-\gamma}$, where the shifting coefficient c is another constant. This property is also called “Mandelbrot law”^[10].

To understand the origins of such Mandelbrot law, Ren, Yang and Wang^[11] proposed a interesting growing network that is generated with linear preferential attachment. In such networks, there exists a recursive dependence relationship between every two consecutive degrees

$$p(k) \left[k + \frac{2 + 2m\beta}{1 - \beta} \right] = p(k - 1) \left[k + \frac{2m\beta}{1 - \beta} - 1 \right] \quad (1)$$

where where $k = 2, \dots, n$, n is the number of nodes. m is a positive integer constant and $\beta \in [0, 1]$ is another constant.

Defining $a = \frac{2m\beta}{1 - \beta} - 1$, $b = \frac{2 + 2m\beta}{1 - \beta}$, we can abbreviate Eq.(1) as

$$p_k [k + b] = p_{k-1} [k + a] \quad (2)$$

To derive the asymptotic of the degree distribution, Ren, Yang and Wang^[11] studied the following three kinds of approximations:

I) forward-difference approximation, assuming

$$\frac{dp(k)}{dk} \approx p(k) - p(k-1) = p(k) - \frac{k+b}{k+a}p(k) = \frac{a-b}{k+a}p(k) \quad (3)$$

we have an estimation of the power-law as

$$p(k) \propto (k+a)^{-(b-a)} \quad (4)$$

II) backward-difference approximation, assuming

$$\frac{dp(k)}{dk} \approx p(k+1) - p(k) = \frac{k+1+a}{k+1+b}p(k) - p(k) = \frac{a-b}{k+1+b}p(k) \quad (5)$$

we have another estimation of the power-law as

$$p(k) \propto (k+b+1)^{-(b-a)} \quad (6)$$

III) Suppose we must have a Mandelbrot law $p(k) \propto (k+c)^{-\gamma}$. As a result, we have $p(k-1) \propto (k-1+c)^{-\gamma}$. Substitute these two approximations in the logarithm type of Eq.(2), we have

$$\ln \frac{k+a}{k+b} = \ln \frac{p(k)}{p(k-1)} = -\gamma \ln(k+c) + \gamma \ln(k-1+c) \quad (7)$$

Rewrite Eq.(7) as

$$\ln \frac{1+a\frac{1}{k}}{1+b\frac{1}{k}} = \gamma \ln \frac{1+(c-1)\frac{1}{k}}{1+c\frac{1}{k}} \quad (8)$$

and apply the second order Taylor expansion of $\frac{1}{k}$ in Eq.(8), we have

$$p(k) \propto \left(k + \frac{b+a+1}{2} \right)^{-(b-a)} \quad (9)$$

All these three estimations indicates that the scaling exponent of the degree distribution should be $-(b-a)$. Simulation results^[11] show that Eq.(9) gives the best approximation accuracy of the empirical distributions. However, we still need a rigorous proof of this interesting finding.

Indeed, further assuming $\sum_{k=1}^n p(k) = 1$, we have the following matrix equation

$$\begin{bmatrix} 2+a & -(2+b) & 0 & \dots & 0 \\ 0 & 3+a & -(3+b) & \dots & 0 \\ & & \dots & \dots & \dots \\ 0 & 0 & \dots & n+a & -(n+b) \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} \begin{bmatrix} p(1) \\ p(2) \\ \dots \\ p(n-1) \\ p(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix} \quad (10)$$

Using Gaussian elimination algorithm, we can directly solve $p(n)$ from Eq.(10) as

$$\begin{aligned} p(n) &= \left[1 + \frac{n+b}{n+a} + \dots + \prod_{j=2}^n \frac{j+b}{j+a} \right]^{-1} \\ &= \left[1 + \sum_{i=2}^n \prod_{j=i}^n \frac{j+b}{j+a} \right]^{-1} \end{aligned} \quad (11)$$

Based on the recursive relationship Eq.(2), for a given n , we have

$$\begin{aligned} p(k) &= p(n) \left(\prod_{j=k+1}^n \frac{j+b}{j+a} \right) = p(n) \left(\frac{\prod_{j=1}^n \frac{j+b}{j+a}}{\prod_{j=1}^k \frac{j+b}{j+a}} \right) \\ &= p(n) \left(\prod_{j=1}^n \frac{j+a}{j+b} \right) \left(\prod_{j=1}^k \frac{j+a}{j+b} \right) \end{aligned} \quad (12)$$

where $k = 1, \dots, n-1$.

It is well known that for Gamma function $\Gamma(z)$, we have $\Gamma(z+1) = z\Gamma(z)$. So, we get

$$(j+b) = \frac{\Gamma(j+1+b)}{\Gamma(j+b)}, \quad (j+a) = \frac{\Gamma(j+1+a)}{\Gamma(j+a)} \quad (13)$$

where $j = 1, \dots, n-1$.

From Eq.(12), we have

$$\begin{aligned} p(k) &= p(n) \left(\prod_{j=1}^n \frac{j+a}{j+b} \right) \left(\prod_{j=1}^k \frac{\Gamma(j+1+a)}{\Gamma(j+a)} \right) \left(\prod_{j=1}^k \frac{\Gamma(j+b)}{\Gamma(j+1+b)} \right) \\ &= p(n) \left(\prod_{j=1}^n \frac{j+a}{j+b} \right) \frac{\Gamma(k+1+a)}{\Gamma(1+a)} \frac{\Gamma(1+b)}{\Gamma(k+1+b)} \\ &= \lambda \cdot \frac{\Gamma(k+1+a)}{\Gamma(k+1+b)} \end{aligned} \quad (14)$$

where $\lambda = p(n) \left(\prod_{j=1}^n \frac{j+a}{j+b} \right) \frac{\Gamma(1+b)}{\Gamma(1+a)}$ is a constant.

Eq.(14) indicates that $p(k)$ has the same asymptotic behavior of $\frac{\Gamma(k+1+a)}{\Gamma(k+1+b)}$. Actually, the quotient of two Gamma functions is a difficult problem that received consistent attentions^[12–15]. There are numbers of approximation formulas which are not accurate enough for the above applications. Fortunately, an important results had been obtained very recently^[15] as

Lemma 1^[15] Given two constants s and t , when $x \rightarrow \infty$, we have

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{\frac{1}{t-s}} \sim \sum_{k=0}^{\infty} F_k(t, s) x^{-n+1} \quad (15)$$

where $F_k(t, s)$ are the polynomials of order n defined by

$$F_0(t, s) = 1 \quad (16)$$

$$F_n(t, s) = \frac{1}{n} \sum_{k=1}^n (-1)^{k+1} \frac{B_{k+1}(t) - B_{k+1}(s)}{(k+1)(t-s)} F_{n-k}(t, s) \quad (17)$$

where $n \geq 1$, $B_k(t)$ is the Bernoulli polynomials (page 40 of [16]) for t .

Based on **Lemma 1**, from Eq.(14), we can have an accurate expansion of the degree distribution as follows

$$\left[\frac{p(k)}{\lambda} \right]^{\frac{1}{a-b}} \sim k + \frac{a+b+1}{2} + \frac{1-(a-b)^2}{24} k^{-1} + \dots \quad (18)$$

As $k \rightarrow \infty$, we have $\left[\frac{p(k)}{\lambda} \right]^{\frac{1}{a-b}} \approx k + \frac{a+b+1}{2}$. Thus, we reach the following conclusion rigorously.

Theorem 1 The degree distribution follows an asymptotic Mandelbrot law Eq.(9) for some complex networks that grow with linear preferential attachment depicted by Eq.(2).

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